Statistical Mechanics of Multipartite Entanglement

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Objectives

- Define maximal multipartite entanglement and look for *maximal multipartite entangled states* (MMES)
- Recast the search for MMES in terms of (classical) statistical mechanics
- Construct potential of multipartite entanglement →cost function
- Contruct partition function of multipartite entanglement

Links and novelty: complex systems, frustration

Example: qubits

$$|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad |1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

Bell (or EPR) state

$$|\eta\rangle = \frac{1}{\sqrt{2}}(|0\rangle_A|1\rangle_B + |1\rangle_A|0\rangle_B)$$

$$\rho = |\eta\rangle\langle\eta| \longrightarrow \rho_A = \mathrm{Tr_B}(\rho) = \begin{pmatrix} 1/2 & 0 \\ 0 & 1/2 \end{pmatrix}$$

Separable State

$$|\eta\rangle = |0\rangle_A|1\rangle_B$$

$$\longrightarrow \quad \rho_A = \mathrm{Tr_B}(\rho) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

Entanglement is "encoded" in the eigenvalues of the density matrix

$$\rho = \begin{pmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ \dots & \dots & \dots & 0 \\ 0 & 0 & 0 & \lambda_N \end{pmatrix}$$

N: dimension of the Hilbert space of the system

λ_i: eigenvalues of ρ

$$L(\rho) = \mathrm{Tr}(\rho^2) = \sum_i \lambda_i^2$$

purity (linear entropy): one measure of entanglement

Remark (profound in its simplicity)

Parisi: complex systems

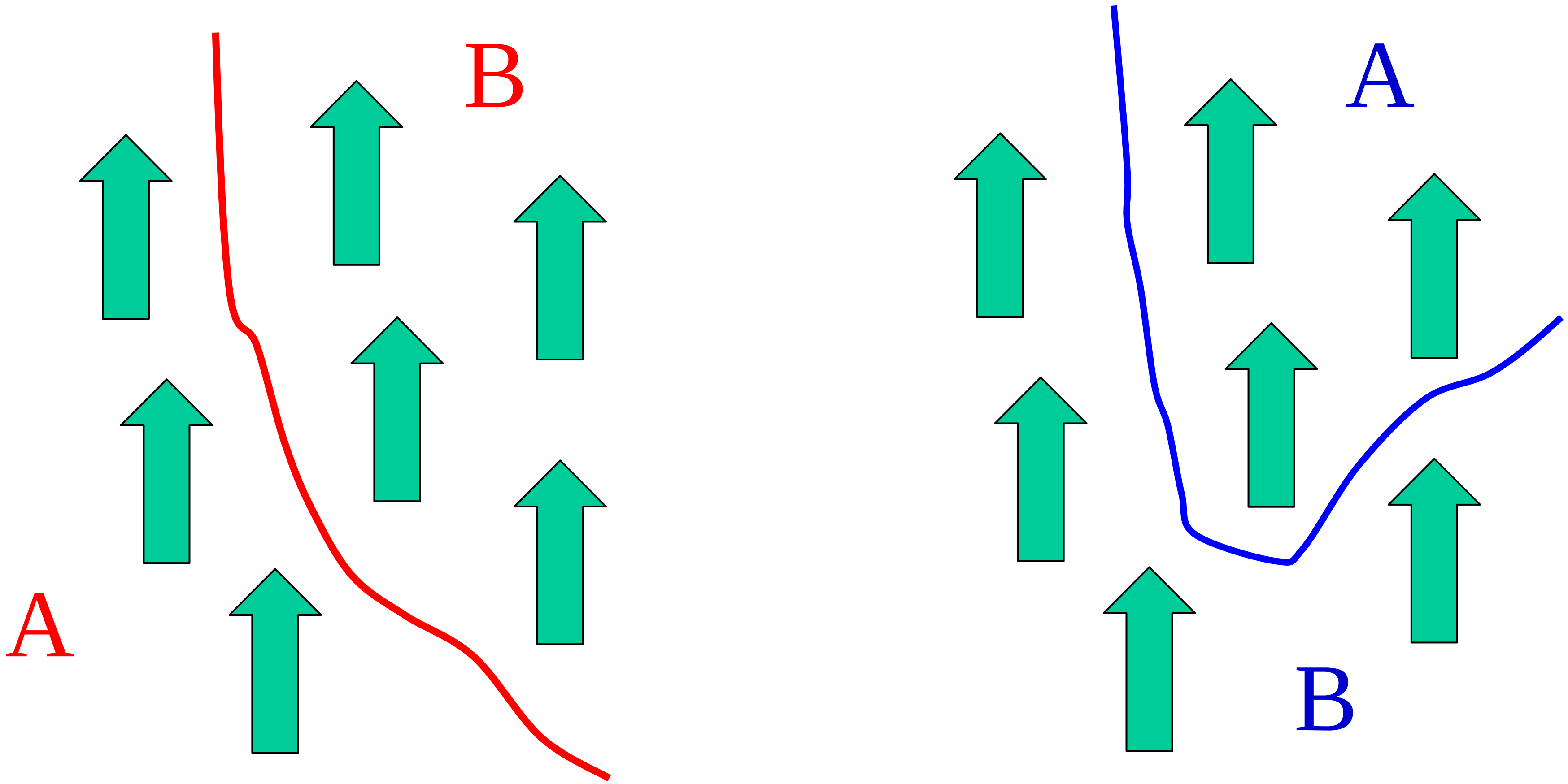
Man’ko, Marmo, Sudarshan, Zaccaria:
multipartite entanglement (J. Phys. A 02-03)

One (or a few) real number(s) is/are not enough

Observation: number of measures (i.e. real numbers) needed to quantify **multipartite** entanglement grows **exponentially** with ***n***=number of qubits

Observation calls for ⇒ statistical methods!

Towards multipartite entanglement



entanglement (e.g. purity) will depend on the bipartition

Main idea

Look at entanglement distribution over all possible (balanced) bipartitions.

Take this distribution as a characterization of multipartite (global) entanglement

multipartite entanglement

V. Coffman, J. Kundu and W. K. Wootters, Phys. Rev. A 61, 052306 (2000)

A. Wong and N. Christensen, Phys. Rev. A 63, 044301 (2001)

D. Bruss, J. Math. Phys. 43, 4237 (2002)

D.A. Meyer and N.R. Wallach, J. Math. Phys. 43, 4273 (2002)

J. Bergou, in preparation

entanglement

W. K. Wootters, Quantum Inf. and Comp. 1, 27 (2001)

L. Amico, R. Fazio, A. Osterloh and V. Vedral Rev. Mod. Phys. 80, 517 (2008)

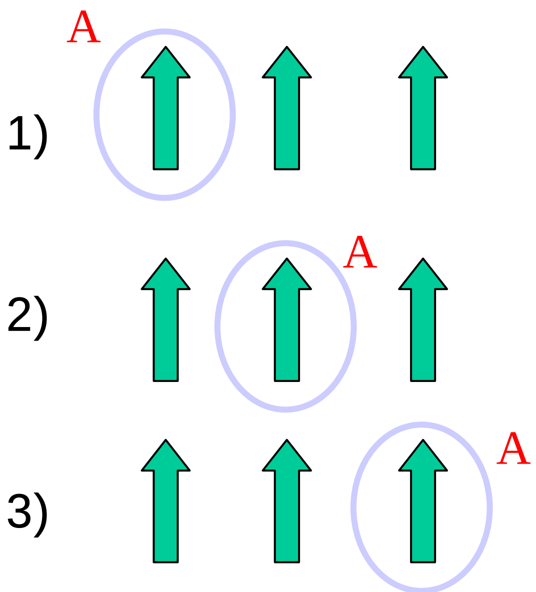
Characterization of entanglement

- The **average** will be a measure of the amount of entanglement in the system, while the **variance** will measure how well such entanglement is distributed: a smaller variance will correspond to a larger insensitivity to the choice of the partition.
- The distribution of purity characterizes entanglement.
- Facchi, Florio, P. [PRA 74, 042331 (2006)]

An example: GHZ

[Greenberger, Horne, Zeilinger (1990)]

$$|\eta\rangle = \frac{1}{\sqrt{2}}(|0\rangle|0\rangle|0\rangle + |1\rangle|1\rangle|1\rangle)$$



$$N_{AB} = \pi_{AB}^{-1} = 2$$

For all bipartitions!!

↓

Entanglement is well distributed

Second example: Ising

$$|\eta\rangle = \frac{1}{\sqrt{2}}\Big(\underbrace{|0\rangle|0\rangle\ldots|0\rangle}_{n \text{ spins}} + \underbrace{|1\rangle|1\rangle\ldots|1\rangle}_{n \text{ spins}}\Big)$$

$$N_{AB} = \pi_{AB}^{-1} = 2$$

For all bipartitions!!

↓

Well distributed entanglement → But small amount of entanglemnt

Typical states

E. Lubkin, J. Math. Phys. 19, 1028 (1978)

S. Lloyd and H. Pagels, Ann. Phys., NY, 188, 186 (1988)

D.N. Page, Phys. Rev. Lett. 71, 1291 (1993)

K. Zyczkowski and H.-J. Sommers J. Phys. A 34, 7111 (2001)

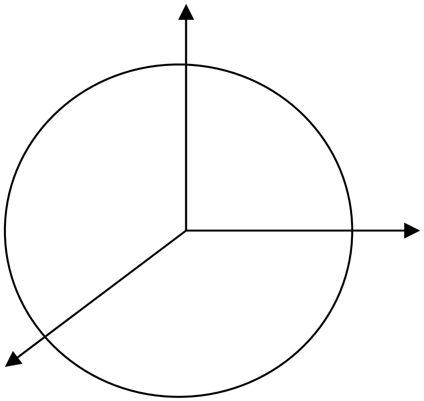
A. J. Scott and C. M. Caves, J. Phys. A: Math. Gen. 36, 9553 (2003)

O. Giraud, J. Phys. A: Math. Gen. 40, F1053 (2007)

$$|\psi\rangle = \sum_{k=1}^{N} r_k e^{i\phi_k} |k\rangle$$

$\phi_k \in [0, 2\pi]$ Independent uniformly distributed random variables

$$\boldsymbol{r} = (r_1, \ldots, r_N)$$



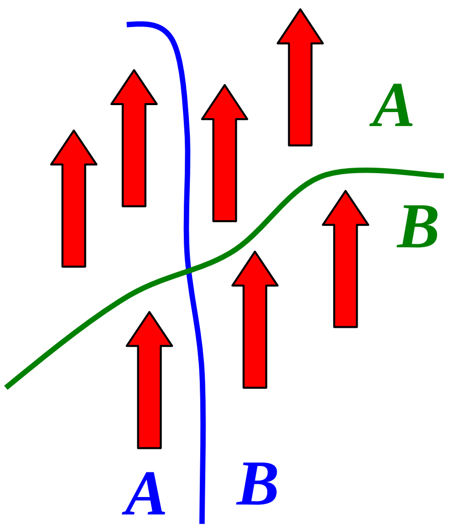
Random point uniformly distributed on the hypersphere $S^{N-1} = \{\boldsymbol{r} \in \mathcal{R}^N | \boldsymbol{r}^2 = 1\}$

distribution: $p(\boldsymbol{r}) = \frac{2^N}{\pi^{N/2}} \Gamma\left(\frac{N}{2}\right) \delta(1 - \boldsymbol{r}^2)$

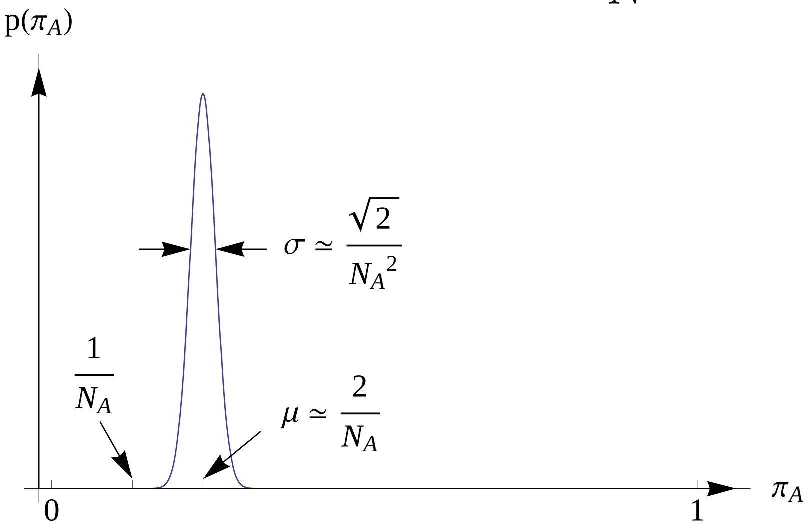
Typical states: in the limit of large number N of spins purity π_{AB} is a Gaussian random variable (Central Limit Theorem) and we can analytically obtain mean and variance of the distribution, when A and B are varied:

$$\nu_{AB} = E[\pi_{AB}] = \frac{N_A + N_B - 1}{N}$$

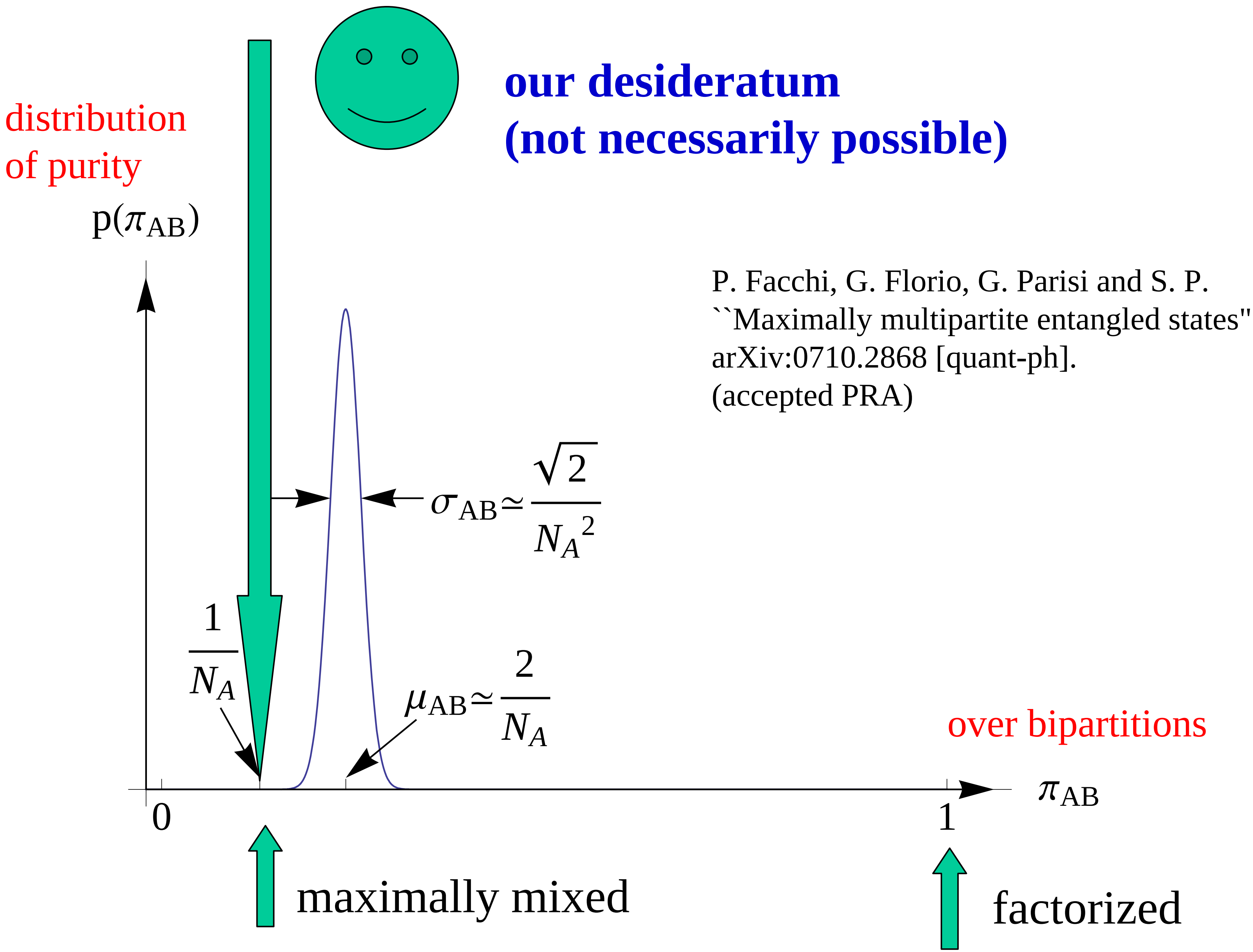
$$\sigma^2_{AB} = E[\pi^2_{AB}] - \nu^2_{AB} = 2\,\frac{(N_A - 1)(N_B - 1)}{N^3}$$



Puritydepends on bipartition π_{AB}



Typical states are *very entangled*.
But do there exist states that are ***more entangled*** ?


our desideratum
(not necessarily possible)
distribution
of purity
$p(\pi_{AB})$
P. Facchi, G. Florio, G. Parisi and S. P.
``Maximally multipartite entangled states''
arXiv:0710.2868 [quant-ph].
(accepted PRA)
$\sigma_{AB} \simeq \frac{\sqrt{2}}{N_A^2}$
$\frac{1}{N_A}$
$\mu_{AB} \simeq \frac{2}{N_A}$
over bipartitions
π_{AB}
0
1
maximally mixed
factorized

N qubits

We consider an ensemble $S = \{1, 2, \ldots, n\}$ of n qubits in the Hilbert space $\mathcal{H} = (\mathbb{C}^2)^{\otimes n}$, whose state is

$$|\psi\rangle = \sum_{k \in Z_2^n} z_k |k\rangle, \quad z_k \in \mathbb{C}, \quad \sum_{k \in Z_2^n} |z_k|^2 = 1,$$

where $k = (k_i)_{i \in S}$, with $k_i \in Z_2 = \{0, 1\}$, and

$$|k\rangle = \bigotimes_{i \in S} |k_i\rangle_i, \qquad |k_i\rangle_i \in \mathbb{C}_i^2.$$

Fix bipartition *(A,A)*

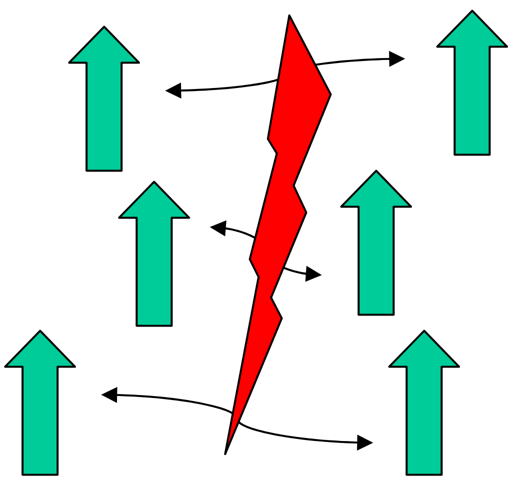
$$|\psi\rangle = \sum_{k \in Z_2^n} z_k |k^A\rangle_A \otimes |k^{\bar{A}}\rangle_{\bar{A}},$$

$$k^A = (k_i)_{i \in A},\ |k^A\rangle_A = \bigotimes_{i \in A} |k_i\rangle_i \in \mathcal{H}_A.$$

purity

$$\pi_A = \sum_{k_i \in Z_2^n} z_{k_1} \bar{z}_{k_2} z_{k_3} \bar{z}_{k_4} \delta_{k_1^A, k_4^A} \delta_{k_2^A, k_3^A} \delta_{k_1^{\bar{A}}, k_2^{\bar{A}}} \delta_{k_3^{\bar{A}}, k_4^{\bar{A}}}$$

Easy to find **maximally bipartite state**
(Schmidt basis)



What about maximal **multi**partite?

Maximally multipartite entangled state (**MMES**):
minimizer of ***potential of multipartite entanglement***

$$\pi_{\mathrm{ME}} = \binom{n}{n_A}^{-1} \sum_{|A|=n_A} \pi_A$$

$$1/N_A \leq \pi_{\mathrm{ME}} \leq 1$$

$n_A = [n/2]$ (balanced bipartitions)

(average) linear entropy $S_L = \frac{N_A}{N_A - 1}(1 - \pi_{\mathrm{ME}})$

J. Scott, Phys. Rev. A 69, 052330 (2004) (was studying self-dual codes)

D.A. Meyer and N.R. Wallach, J. Math. Phys. 43, 4273 (2002)

G. Rigolin, T. R. de Oliveira and M. C. de Oliveira, Phys. Rev. A 74, 022314 (2006)

More general cost function

$$\tilde{\pi}_{\mathrm{ME}}(\lambda) = \pi_{\mathrm{ME}} + \lambda \sigma_{\mathrm{ME}},$$

where $\lambda \geq 0$ is a Lagrange multiplier and

$$\sigma^2_{\mathrm{ME}} = \begin{pmatrix} n \\ n_A \end{pmatrix}^{-1} \sum_{|A|=n_A} \left(\pi_A - \pi_{\mathrm{ME}}\right)^2$$

One finds

$$\pi_{\mathrm{ME}} = \frac{1}{|\mathcal{P}_n|} \sum_{p \in \mathcal{P}_n} \pi_{p(C)} = \sum_{k_i \in Z_2^n} \Delta(k_1, k_2, k_3, k_4) z_{k_1} \bar{z}_{k_2} z_{k_3} \bar{z}_{k_4}$$

$$\begin{aligned}
\Delta(k_1, k_2, k_3, k_4) &= \frac{1}{n!} \sum_{p \in \mathcal{P}_n} \delta_{k_1^{p(C)}, k_4^{p(C)}} \delta_{k_2^{p(C)}, k_3^{p(C)}} \delta_{k_1^{p(\bar{C})}, k_2^{p(\bar{C})}} \delta_{k_3^{p(\bar{C})}, k_4^{p(\bar{C})}} \\
&= g(k_1 \oplus k_4 \vee k_2 \oplus k_3, k_1 \oplus k_2 \vee k_3 \oplus k_4) \\
g(a, b) &= \frac{1}{|\mathcal{P}_n^{n_A}|} \delta_{a \wedge b, 0} \begin{pmatrix} n - |a| - |b| \\ n_A - |b| \end{pmatrix}
\end{aligned}$$

where $a \oplus b = (a_i + b_i \bmod 2)_{i \in S}$ XOR

$a \vee b = (a_i + b_i - a_i b_i)_{i \in S}$ OR

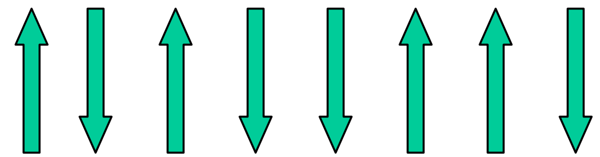
$a \wedge b = (a_i b_i)_{i \in S}$ AND

and $|a| = \sum_{i \in S} a_i$.

For example, for 2 qubits one gets $\pi_{\mathrm{ME}}^{(2)} = 1 - 2|z_0 z_3 - z_1 z_2|^2$

Particular class of states

$k = 0 \quad 1 \quad 0 \quad 1 \quad 1 \quad 0 \quad 0 \quad 1$



$$|\psi\rangle = \frac{1}{\sqrt{N}} \sum_{k \in Z_2^n} e^{i\varphi_k} |k\rangle$$

Bipartition C = permutation $p = \; 0 \quad\quad\quad 1 \quad 1 \quad\quad\quad 1$

One finds

$$\pi_{p(C)} = \frac{N_A + N_{\bar{A}} - 1}{N} + \frac{4}{N^2} \sum_{l<l', m<m'} \cos\left(\varphi^p_{lm} - \varphi^p_{l'm} + \varphi^p_{l'm'} - \varphi^p_{lm'}\right)$$

$$1/N_A \leq \pi_{\mathrm{ME}} \leq 1$$

maybe not surprisingly (complexity)

bears the symptoms of frustration

2 qubits

$$\pi_{\mathrm{ME}}^{(2)} = \frac{3}{4} + \frac{1}{4}\cos\left(\varphi_0 - \varphi_1 - \varphi_2 + \varphi_3\right)$$

$$\pi_{\mathrm{ME}}^{(2)} = 1/2 \quad \Longrightarrow \quad \varphi_0 - \varphi_1 - \varphi_2 + \varphi_3 = \pi$$

$$|\psi_2\rangle = \frac{1}{2}\left(e^{i\varphi_0}|0\rangle + e^{i\varphi_1}|1\rangle + e^{i\varphi_2}|2\rangle - e^{i(-\varphi_0+\varphi_1+\varphi_2)}|3\rangle\right)$$

MMES: Bell state

3 qubits

$$\pi_{\mathrm{ME}}^{(3)} = \frac{5}{8} + \frac{1}{48}\sum_{p}\Big[\cos\big(\varphi_{p(0)} + \varphi_{p(7)} - \varphi_{p(1)} - \varphi_{p(6)}\big)$$
$$+ \cos\big(\varphi_{p(2)} + \varphi_{p(5)} - \varphi_{p(4)} - \varphi_{p(3)}\big)$$
$$+ 2\cos\big(\varphi_{p(0)} + \varphi_{p(3)} - \varphi_{p(1)} - \varphi_{p(2)}\big)$$
$$+ 2\cos\big(\varphi_{p(7)} + \varphi_{p(4)} - \varphi_{p(6)} - \varphi_{p(5)}\big)\Big],$$

$$|\psi_3\rangle = \frac{1}{8}(e^{i\varphi_0}|0\rangle + e^{i\varphi_1}|1\rangle + e^{i\varphi_2}|2\rangle - e^{i(-\varphi_0+\varphi_1+\varphi_2)}|3\rangle$$
$$+ e^{i\varphi_4}|4\rangle - e^{i(-\varphi_0+\varphi_1+\varphi_4)}|5\rangle + e^{-i\varphi_6}|6\rangle$$
$$+ e^{i(-\varphi_0+\varphi_1+\varphi_6)}|7\rangle)$$

MMES: GHZ particular case **(clear symptoms of frustration)**

4 qubit: NO ?!?

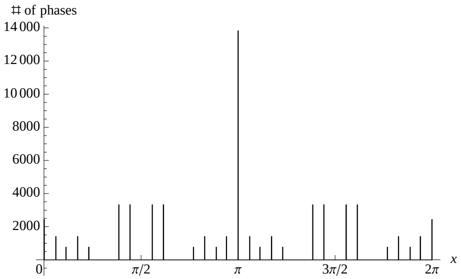
A. Higuchi and A. Sudbery, Phys. Lett. A 273, 213 (2000)
S. Brierley and A. Higuchi, J. Phys. A: Math. Theor. 40, 8455 (2007)
A. Borras, A.R. Plastino, J. Batle, C. Zander, M. Casas, and A. Plastino, J. Phys. A: Math. Theor. 40, 13407 (2007).

Theorem: minimum 1/4 *cannot* be reached
(one gets 1/3 + numerics + stationary + local min)

5 qubits: OK

$$(\varphi_k) = (0,0,0,0,0,\pi,\pi,0,0,\pi,\pi,0,0,0,0,0,\\ 0,0,\pi,\pi,0,\pi,0,\pi,\pi,0,\pi,0,\pi,\pi,0,0)$$

6 qubits: OK



$$|\psi\rangle = \frac{1}{\sqrt{N}} \sum_{k\in Z_2^n} e^{i\varphi_k} |k\rangle$$

7 qubits: NO ?

[A. J. Scott, Phys. Rev. A 69, 052330 (2004)]

> 7 qubits: NO

[see also A. J. Scott 2004 and E. Rains, IEEE 1998-99]

in conclusion:

- n=2 → "perfect" MMES exist
- n=3 → "perfect" MMES exist
- n=4 → "perfect" MMES do **not** exist
- n=5 → "perfect" MMES exist
- n=6 → "perfect" MMES exist
- n=7 → not clear: 0.134 *vs* 0.125
- n>7 → "perfect" MMES do **not** exist

Requirement purity = 1/N for *all balanced bipartitions* is impossible.
Different bipartitions "compete" with each other:

frustration

Statistical mechanics of multipartite entanglement

Partition function: $Z(\beta) = \int d\mu_C(z) e^{-\beta H(z)}$

fictitious temperature

H = average purity over bipartitions

$$|\psi\rangle = \sum_{j=0}^{N-1} z_j |j\rangle$$

$$d\mu_C(z) = \prod_k dz_k d\bar{z}_k \delta\left(1 - \sum_k |z_k|^2\right)$$

(measure of typical states)

Facchi, Florio, Marzolino, Parisi, P., arXiv:0803.4498 [quant-ph]

High, low and negative temperature limits

$$Z(\beta) = \int d\mu_C(z) e^{-\beta H(z)}$$

Energy (purity) distribution function

$$P_\beta(E) = \frac{1}{Z(\beta)} \int d\mu_C(z) \delta\left(H - E\right) e^{-\beta H}$$

$$= \frac{e^{-\beta E} P_0(E)}{\int_{E_0}^{1} dE e^{-\beta E} P_0(E)},$$

Average energy (purity)

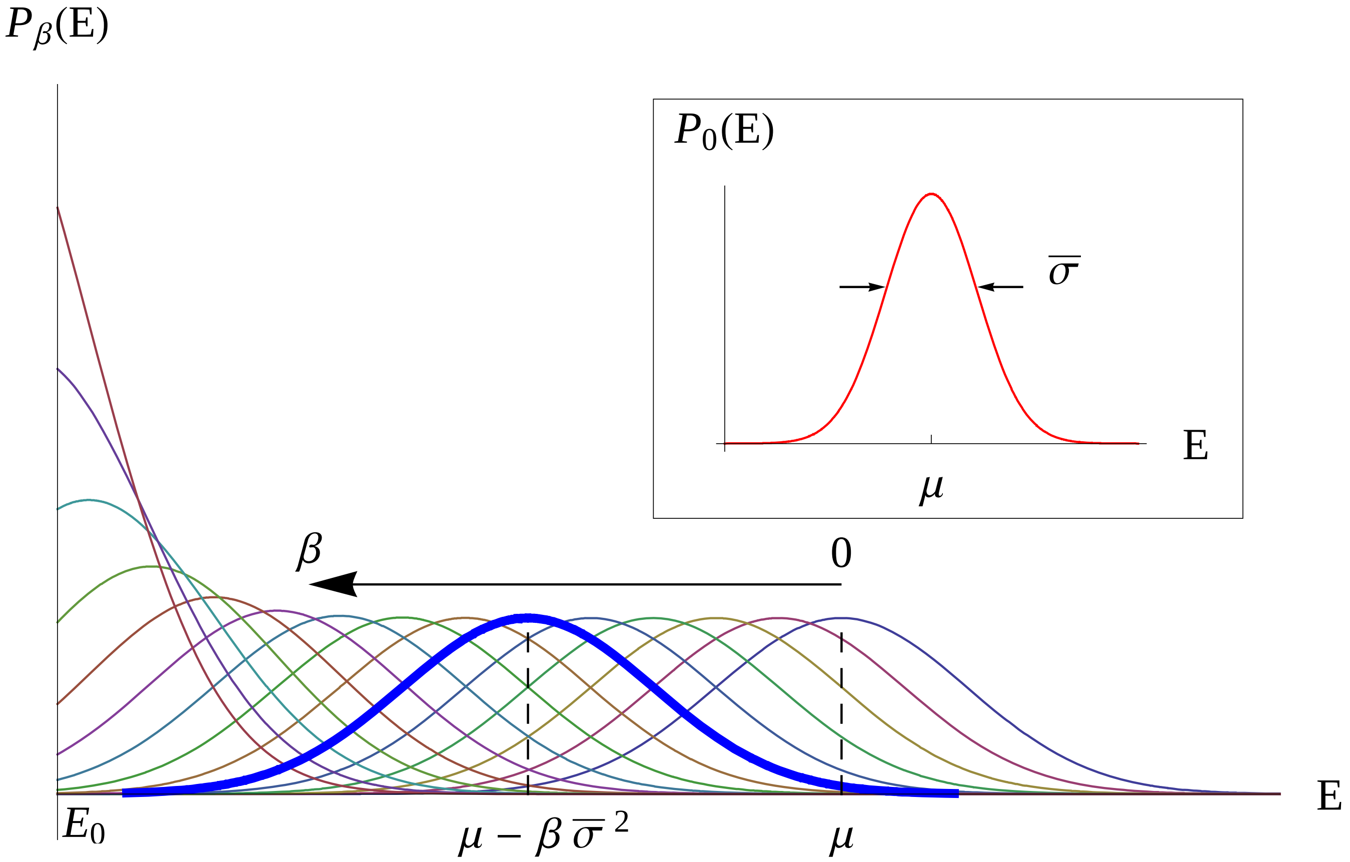
$$\langle H \rangle_\beta = \frac{1}{Z(\beta)} \int d\mu_C(z) H e^{-\beta H}$$

$$= \int_{E_0}^{1} dE E P_\beta(E) = -\frac{\partial}{\partial \beta} \ln Z(\beta)$$

First results:

$\beta \to +\infty$	$H = E_0$ (min)	MMES
$\beta \to 0$	$H \simeq \mu$	typical states
$\beta \to -\infty$	$H = 1$ (max)	separable states

Behaviour at fixed N

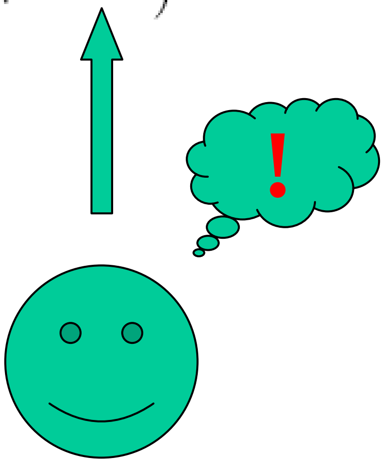


Thermodynamical limit

cumulants $\kappa_\beta^{(m)}(H) = (-)^{m-1} \frac{\partial^{m-1}}{\partial \beta^{m-1}} \langle H \rangle_\beta$

second cumulant (standard deviation) exact:

$$\bar{\sigma}^2 = \kappa_0^{(2)}(H) \sim 3\sqrt{2} N^{-4+\log_2 3} \simeq O(N^{-2.42})$$



Perspectives

Few qubit applications (new!)

Cryptography (new!)

More general applications?

Links with complexity and frustration (in progress)

Hope for characterization of large entanglement (statistical mechanics + complex systems)